Examples:

$$\dfrac{\dfrac{(1)\ \phi \vdash * : \square \quad \text{(sort)}}{(2)\ \alpha : * \vdash \alpha : * \quad \text{(var)}} \qquad \dfrac{(1)\ \phi \vdash * : \square \quad \text{(sort)}}{(2)\ \alpha : * \vdash \alpha : * \quad \text{(var)}}}{(4)\ \alpha : *,\ x : \alpha \vdash \alpha : * \quad \text{(weak)}}$$

$$\dfrac{\dfrac{(1)\ \phi \vdash * : \square \quad \text{(sort)}}{(2)\ \alpha : * \vdash \alpha : *} \text{(var)} \qquad \dfrac{(1)\ \phi \vdash * : \square \ \text{(sort)} \quad (1)\ \phi \vdash * : \square \ \text{(sort)}}{(5)\ \alpha : * \vdash * : \square \quad \text{(weak)}}}{(6)\ \alpha : *,\ \beta : * \vdash \alpha : * \quad \text{(weak)}}$$

$$\dfrac{\dfrac{(1)\ \phi \vdash * : \square \ \text{(sort)} \quad (1)\ \phi \vdash * : \square \ \text{(sort)}}{(5)\ \alpha : * \vdash * : \square \quad \text{(weak)}}}{(7)\ \alpha : *,\ \beta : * \vdash \beta : * \quad \text{(var)}}$$

Tree format:

(1)	$* : \square$	(sort)
	$\boxed{\alpha : *}$	
(2)	$\alpha : *$	(var) on (1)
	$\boxed{x : \alpha}$	
(3)	$x : \alpha$	(var) on (2)
(4)	$\alpha : *$	(weak) on (2) and (2)
(5)	$* : \square$	(weak) on (1) and (1)
	$\boxed{\beta : *}$	
(6)	$\alpha : *$	(weak) on (2) and (5)
(7)	$\beta : *$	(var) on (5)

The formation rule (form)

$$\text{(form)}\ \dfrac{\Gamma \vdash A : s \qquad \Gamma \vdash B : s}{\Gamma \vdash A \to B : s}$$

Example:

⋮

(8)	$\alpha \to \beta : *$	(form) on (6) and (7)
(9)	$* \to * : \square$	(form) on (5) and (5)

The application and abstraction rules

$$\text{(appl)}\ \dfrac{\Gamma \vdash M : A \to B \ ,\ \Gamma \vdash N : A}{\Gamma \vdash MN : B}$$

$$\text{(abst)}\ \dfrac{\Gamma, x : A \vdash M : B \qquad \Gamma : A \to B : s}{\Gamma \vdash \lambda x : A . M : A \to B}$$

	$\boxed{\beta : *}$	
(10)	$* : \square$	(weak) on (1) and (1)
	$\boxed{\alpha : *}$	
(11)	$\alpha : *$	(var) on (10)
(12)	$\alpha \to \alpha : *$	(form) on (11) and (11)
(13)	$* \to * : \square$	(form) on (10) and (10)
(14)	$\lambda \alpha : * . \alpha \to \alpha : * \to *$	(abst) on (12) and (13)
(15)	$\beta : *$	(var) on (1)
(16)	$(\lambda \alpha : * . \alpha \to \alpha)\beta : *$	(appl) on (14) and (15)
(17)	$\beta \to \beta : *$	(form) on (15) and (15)

$(\lambda \alpha : * . \alpha \to \alpha)\beta \to_\beta \beta \to \beta$

Shortened derivations

Silently execute (sort), (var), (weak) and (form) rules, second premiss of (abst) rule

(a)	$\boxed{\beta : *}$	
(b)	$\boxed{\alpha : *}$	necessary for (14)
(12)	$\alpha \to \alpha : *$	(form) on (b) and (b)
(14)	$\lambda \alpha : * . \alpha \to \alpha : * \to *$	(abst) on (12) ← first premiss only
(16)	$(\lambda \alpha : * . \alpha \to \alpha)\beta : *$	(appl) on (14) and (a)

The conversion rule

Problem: $(\lambda \alpha : * . \alpha \to \alpha)\beta \to_\beta \beta \to \beta$

$\beta : *, x : (\lambda \alpha : * . \alpha \to \alpha)\beta \vdash x : (\lambda \alpha : * . \alpha \to \alpha)\beta$ (var) on (16)

naturally, we also want

$\beta : *, x : (\lambda \alpha : * . \alpha \to \alpha)\beta \vdash x : \beta \to \beta$

but this cannot be derived so far!

Solution: thus B is already well-formed; but B' may be not, even if $B =_\beta B'$:

$$(conv) \quad \frac{\Gamma \vdash A : B \qquad \Gamma \vdash B' : s}{\Gamma \vdash A : B'} \quad \text{if } B =_\beta B'$$

$\beta \to \gamma =_\beta (\lambda \alpha : * . \beta \to \gamma) M$ holds for every term M, but $(\lambda \alpha : * . \beta \to \gamma) M$ need not be well-formed (if M is not)

Example:

⋮

	$\boxed{x : (\lambda \alpha : * . \alpha \to \alpha)\beta}$	
(18)	$x : (\lambda \alpha : * . \alpha \to \alpha)\beta$	(var) on (16)
(19)	$\beta \to \beta : *$	(form) on (15) and (15)
(20)	$x : \beta \to \beta$	(conv) on (18) and (19)

Shortened derivation: we silently execute the second premiss of the (conv) rule

⋮

	$\boxed{x : (\lambda \alpha : * . \alpha \to \alpha)\beta}$	
(18)	$x : (\lambda \alpha : * . \alpha \to \alpha)\beta$	(var) on (16)
(20)	$x : \beta \to \beta$	(conv) on (18)

Subject reduction, type reduction, conversion

$$\frac{\begin{array}{c}\Gamma \vdash A : B \\ \downarrow \beta \\ A'\end{array}}{\Gamma \vdash A' : B}$$

← subject reduction (theorem)

$$\frac{\begin{array}{c}\Gamma \vdash A : B \\ \qquad\ \downarrow \beta \\ \qquad\ B'\end{array}}{\Gamma \vdash A : B'}$$

← type reduction (subcase of (conv)) (also not derivable without (conv) rule)

$$\frac{\begin{array}{c}\Gamma \vdash A : B \\ \qquad =_\beta \\ \qquad B'\end{array}}{\Gamma \vdash A : B'} \quad \text{if } \Gamma \vdash B' : s$$

← conversion ((conv) rule)

Properties of $\lambda\underline{\omega}$

$\lambda\underline{\omega}$ satisfies all theorems that $\lambda 2$ does, with a small modification to:

- Uniqueness of types lemma: if $\Gamma \vdash A : B$ and $\Gamma \vdash A : B'$, then $B =_\beta B'$ (instead of $B \equiv B'$)